

# A REMARK ON LOW REGULARITY SOLUTIONS OF THE CHERN-SIMONS-DIRAC SYSTEM

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ABSTRACT. An alternative proof of low regularity well-posedness for the Chern-Simons-Dirac system in Coulomb gauge is given which completely avoids the use of any null structure similarly to a recent result of Bournaveas-Candy-Machihara. An unconditional uniqueness result is also given.

## 1. INTRODUCTION AND MAIN RESULTS

Consider the Chern-Simons-Dirac system in two space dimensions :

$$\frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = -J^\mu \quad (1)$$

$$i\gamma^\mu D_\mu \psi = m\psi, \quad (2)$$

with initial data

$$A_\mu(0) = a_\mu, \quad \psi(0) = \psi_0, \quad (3)$$

where we use the convention that repeated upper and lower indices are summed, Greek indices run over 0,1,2 and Latin indices over 1,2. Here

$$\begin{aligned} D^\mu &:= \partial_\mu - iA_\mu \\ F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu \\ J^\mu &:= \langle \gamma^0 \gamma^\mu \psi, \psi \rangle \end{aligned}$$

Here  $F_{\mu\nu} : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$  denotes the curvature,  $\psi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}^2$ , and  $A_\mu : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$  the gauge potentials. We use the notation  $\langle \cdot, \cdot \rangle$  for the inner product in  $\mathbb{C}^2$ ,  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ , where we write  $(x^0, x^1, \dots, x^n) = (t, x^1, \dots, x^n)$  and also  $\partial_0 = \partial_t$  and  $\nabla = (\partial_1, \partial_2)$ .  $\epsilon^{\mu\nu\rho}$  is the totally skew-symmetric tensor with  $\epsilon^{012} = 1$ , and  $m \geq 0$ .  $\gamma^\mu$  are the Pauli matrices  $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ ,  $\gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

The equations are invariant under the gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi, \quad \psi \rightarrow \psi' = e^{i\chi} \psi, \quad D_\mu \rightarrow D'_\mu = \partial_\mu - iA'_\mu.$$

The most common gauges are the Coulomb gauge  $\partial^j A_j = 0$ , the Lorenz gauge  $\partial^\mu A_\mu = 0$  and the temporal gauge  $A_0 = 0$ .

Our main aim is to give a simple proof of local well-posedness for data  $\psi_0 \in H^s(\mathbb{R}^2)$  for  $s > \frac{1}{4}$ , especially we want to show that no null condition is necessary in the case of the Coulomb gauge. Critical with respect to scaling is the case  $s = 0$ . The same result was proven recently in Coulomb as well as Lorenz gauge by M. Okamoto [8] using a null structure of the system. Earlier results were given by Huh

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[5] in the Lorenz gauge for data  $\psi_0 \in H^{\frac{5}{8}}$ ,  $a_\mu \in H^{\frac{1}{2}}$  using a null structure, in the Coulomb gauge for  $\psi_0 \in H^{\frac{1}{2}+\epsilon}$ ,  $a_i \in L^2$ , and in temporal gauge for  $\psi_0 \in H^{\frac{3}{4}+\epsilon}$ ,  $a_j \in H^{\frac{3}{4}+\epsilon} + L^2$ , both without using a null structure. Huh-Oh [6] proved local well-posedness in Lorenz gauge for  $\psi_0 \in H^s$ ,  $a_\mu \in H^s$  for  $s > \frac{1}{4}$  also making use of a null structure. The methods of Okamoto and Huh-Oh are different. Okamoto reduces the problem to a single Dirac equation with cubic nonlinearity for  $\psi$ , which does not contain  $A_\mu$  any longer. From a solution  $\psi$  of this equation the potentials  $A_\mu$  can be constructed by solving a wave equation in Lorenz gauge and an elliptic equation in Coulomb gauge. Our proof also relies on this approach. Huh-Oh on the other hand directly solve a coupled system of a Dirac equation for  $\psi$  and a wave equation for  $A_\mu$ .

Our result shows local well-posedness for data  $\psi_0 \in H^s$  down to  $s = \frac{1}{4} +$  without use of the null structure. Therefore it works also for more general systems which lead to cubic Dirac equations of the symbolic form

$$(-i\alpha^\mu \partial_\mu + m\beta)\psi \sim \nabla^{-1} \langle \psi, \psi \rangle \psi.$$

An almost identical result was recently given by Bournaveas-Candy-Machihara [4] who were also able to avoid any use of the null structure of the system. Their proof relies on a bilinear Strichartz estimate given by Klainerman-Tataru [7] whereas we make use of bilinear estimates in wave-Sobolev spaces given by d'Ancona-Foschi-Selberg [2].

Our result gives uniqueness in a certain subspace of  $C^0([0, T], H^s)$  of  $X^{s,b}$ -type. Thus it is natural to consider the question whether unconditional uniqueness also holds, namely in  $C^0([0, T], H^s)$ . We give a positive answer if  $s > \frac{1}{3}$  using an idea of Zhou [10].

We exclusively study the Coulomb gauge condition  $\partial_j A^j = 0$ . In this case one easily checks using (1) that the potentials  $A_\mu$  satisfy the elliptic equations

$$A_0 = \Delta^{-1}(\partial_2 J_1 - \partial_1 J_2), \quad A_1 = \Delta^{-1} \partial_2 J_0, \quad A_2 = -\Delta^{-1} \partial_1 J_0. \quad (4)$$

Inserting this into (2) and defining the matrices  $\alpha^\mu = \gamma^0 \gamma^\mu$ ,  $\beta = \gamma_0$  we obtain

$$(i\alpha^\mu \partial_\mu - m\beta)\psi = N(\psi, \psi, \psi), \quad (5)$$

where

$$\begin{aligned} & N(\psi_1, \psi_2, \psi_3) \\ &= \Delta^{-1} (\partial_2 \langle \alpha_1 \psi_1, \psi_2 \rangle - \partial_1 \langle \alpha_2 \psi_1, \psi_2 \rangle + \partial_2 \langle \psi_1, \psi_2 \rangle \alpha_1 - \partial_1 \langle \psi_1, \psi_2 \rangle \alpha_2) \psi_3. \end{aligned}$$

In the sequel we consider this nonlinear Dirac equation with initial condition

$$\psi(0) = \psi_0. \quad (6)$$

Using an idea of d'Ancona - Foschi - Selberg [1] we simplify (5) by considering the projections onto the one-dimensional eigenspaces of the operator  $-i\alpha \cdot \nabla = -i\alpha^j \partial_j$  belonging to the eigenvalues  $\pm|\xi|$ . These projections are given by  $\Pi_\pm = \Pi_\pm(D)$ , where  $D = \frac{\nabla}{i}$  and  $\Pi_\pm(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha)$ . Then  $-i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D)$  and  $\Pi_\pm(\xi)\beta = \beta\Pi_\mp(\xi)$ . Defining  $\psi_\pm := \Pi_\pm(D)\psi$ , the Dirac equation can be rewritten as

$$(-i\partial_t \pm |D|)\psi_\pm = m\beta\psi_\mp + \Pi_\pm N(\psi_+ + \psi_-, \psi_+ + \psi_-, \psi_+ + \psi_-). \quad (7)$$

The initial condition is transformed into

$$\psi_\pm(0) = \Pi_\pm \psi_0. \quad (8)$$

We use the following function spaces and notation.  $H_p^s$  denotes the standard  $L^p$ -based Sobolev space of order  $s$ ,  $H^s = H_2^s$ , and  $B_{p,q}^s$  the Besov space as defined e.g. in [3]. Let  $\hat{\cdot}$  denote the Fourier transform with respect to space and time. The

standard spaces of Bougain-Klainerman-Machedon type  $X_{\pm}^{s,b}$  belonging to the half waves are defined by the completion of  $\mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$  with respect to

$$\|f\|_{X_{\pm}^{s,b}} = \|\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \widehat{f}(\tau, \xi)\|_{L^2},$$

where  $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ .  $X_{\pm}^{s,b}[0, T]$  is the space of restrictions to the time interval  $[0, T]$ . Similarly  $H^{s,b}$  denotes the completion of  $\mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$  with respect to

$$\|f\|_{H^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widehat{f}(\tau, \xi)\|_{L^2}$$

and  $H^{s,b}[0, T]$  its restriction to the time interval  $[0, T]$ .

We remark the embedding  $X_{\pm}^{s,b} \subset H^{s,b}$  for  $b \geq 0$ .

Finally  $a+$  and  $a-$  denote numbers which are slightly larger and smaller than  $a$  respectively, such that  $a-- < a- < a < a+ < a++$ .

We now formulate our results.

**Theorem 1.1.** *Assume  $\psi_0 \in H^s(\mathbb{R}^2)$  with  $s > \frac{1}{4}$ . Then (5),(6) is locally well-posed in  $H^s(\mathbb{R}^2)$ . More precisely there are  $T > 0$ ,  $b > \frac{1}{2}$  such that there exists a unique solution  $\psi = \psi_+ + \psi_-$  with  $\psi_{\pm} \in X_{\pm}^{s,b}[0, T]$ . This solution belongs to  $C^0([0, T], H^s(\mathbb{R}^2))$ .*

The unconditional uniqueness result is the following

**Theorem 1.2.** *Assume  $\psi_0 \in H^s(\mathbb{R}^2)$  with  $s > \frac{1}{3}$ . The solution of (5),(6) is unique in  $C^0([0, T], H^s(\mathbb{R}^2))$ .*

Fundamental for their proof are the following bilinear estimates in wave-Sobolev spaces which were proven by d'Ancona, Foschi and Selberg in the two dimensional case  $n = 2$  in [2] in a more general form which include many limit cases which we do not need.

**Theorem 1.3.** *Let  $n = 2$ . The estimate*

$$\|uv\|_{H^{-s_0, -b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}$$

*holds, provided the following conditions hold:*

$$\begin{aligned} b_0 + b_1 + b_2 &> \frac{1}{2} \\ b_0 + b_1 &> 0 \\ b_0 + b_2 &> 0 \\ b_1 + b_2 &> 0 \\ s_0 + s_1 + s_2 &> \frac{3}{2} - (b_0 + b_1 + b_2) \\ s_0 + s_1 + s_2 &> 1 - (b_0 + b_1) \\ s_0 + s_1 + s_2 &> 1 - (b_0 + b_2) \\ s_0 + s_1 + s_2 &> 1 - (b_1 + b_2) \\ s_0 + s_1 + s_2 &> \frac{1}{2} - b_0 \\ s_0 + s_1 + s_2 &> \frac{1}{2} - b_1 \\ s_0 + s_1 + s_2 &> \frac{1}{2} - b_2 \\ s_0 + s_1 + s_2 &> \frac{3}{4} \end{aligned}$$

$$\begin{aligned}
(s_0 + b_0) + 2s_1 + 2s_2 &> 1 \\
2s_0 + (s_1 + b_1) + 2s_2 &> 1 \\
2s_0 + 2s_1 + (s_2 + b_2) &> 1 \\
s_1 + s_2 &\geq \max(0, -b_0) \\
s_0 + s_2 &> \max(0, -b_1) \\
s_0 + s_1 &> \max(0, -b_2)
\end{aligned}$$

## 2. PROOF OF THE THEOREMS

*Proof of Theorem 1.1.* By standard arguments we only have to show

$$\|N(\psi_1, \psi_2, \psi_3)\|_{X_{\pm 4}^{s, -\frac{1}{2}++}} \lesssim \prod_{i=1}^3 \|\psi_i\|_{X_{\pm i}^{s, \frac{1}{2}+}},$$

where  $\pm_i$  ( $i = 1, 2, 3, 4$ ) denote independent signs.

By duality this is reduced to the estimates

$$J := \int \langle N(\psi_1, \psi_2, \psi_3), \psi_4 \rangle dt dx \lesssim \prod_{i=1}^3 \|\psi_i\|_{X_{\pm i}^{s, \frac{1}{2}+}} \|\psi_4\|_{X_{\pm 4}^{-s, \frac{1}{2}-}}.$$

By Fourier-Plancherel we obtain

$$J = \int_* q(\xi_1, \dots, \xi_4) \prod_{j=1}^4 \widehat{\psi}_j(\xi_j, \tau_j) d\xi_1 d\tau_1 \dots d\xi_4 d\tau_4,$$

where  $*$  denotes integration over  $\xi_1 - \xi_2 = \xi_4 - \xi_3 =: \xi_0$  and  $\tau_1 - \tau_2 = \tau_4 - \tau_3$  and

$$\begin{aligned}
q = \frac{1}{|\xi_0|^2} &[(\xi_0 (\langle \alpha_1 \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \widehat{\psi}_3, \widehat{\psi}_4 \rangle + \langle \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \alpha_1 \widehat{\psi}_3, \widehat{\psi}_4 \rangle) \\
&- \xi_0 (\langle \alpha_2 \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \widehat{\psi}_3, \widehat{\psi}_4 \rangle + \langle \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \alpha_2 \widehat{\psi}_3, \widehat{\psi}_4 \rangle)].
\end{aligned}$$

The specific structure of this term, namely the form of the matrices  $\alpha_j$  plays no role in the following, thus the null structure is completely ignored.

We first consider the case  $|\xi_0| \leq 1$ . In this case we estimate  $J$  as follows:

$$\begin{aligned}
\|\langle \nabla \rangle^{-s-1} |\nabla|^{-\frac{1}{2}} \langle \alpha_i \psi_1, \psi_2 \rangle\|_{L_{xt}^2} &\lesssim \|\langle \nabla \rangle^{-s-1} \langle \alpha_i \psi_1, \psi_2 \rangle\|_{L_x^2 L_x^{\frac{4}{3}}} \lesssim \|\langle \alpha_i \psi_1, \psi_2 \rangle\|_{L_t^2 B_{\frac{4}{3},1}^{-s-1}} \\
&\lesssim \|\langle \alpha_i \psi_1, \psi_2 \rangle\|_{L_t^2 B_{\frac{4}{3},\infty}^{-s-1}} \lesssim \|\langle \alpha_i \psi_1, \psi_2 \rangle\|_{L_t^2 B_{1,\infty}^{-s}} \lesssim \|\psi_1\|_{L_t^4 H_x^s} \|\psi_2\|_{L_t^4 H_x^{-s}},
\end{aligned}$$

where we used the embeddings  $B_{1,\infty}^{-s} \subset B_{\frac{4}{3},\infty}^{-s-1} \subset B_{\frac{4}{3},1}^{-s-1} \subset H_{\frac{4}{3}}^{-s-1}$ , which hold by [3], Thm. 6.2.4 and Thm. 6.5.1. The last inequality follows from [9], namely the Lemma in Chapter 4.4.3. The same estimate holds for  $\alpha_i = I$ . Similarly we obtain

$$\|\langle \nabla \rangle^{-s-1} |\nabla|^{-\frac{1}{2}} \langle \alpha_i \psi_3, \psi_4 \rangle\|_{L_{xt}^2} \lesssim \|\psi_3\|_{L_t^4 H_x^s} \|\psi_4\|_{L_t^4 H_x^{-s}}$$

for arbitrary matrices  $\alpha_i$ , so that we obtain

$$J \lesssim \|\psi_1\|_{X_{\pm 1}^{s, \frac{1}{4}}} \|\psi_2\|_{X_{\pm 2}^{-s, \frac{1}{4}}} \|\psi_3\|_{X_{\pm 3}^{s, \frac{1}{4}}} \|\psi_4\|_{X_{\pm 4}^{-s, \frac{1}{4}}},$$

which is more than enough. From now on we assume  $|\xi_0| \geq 1$ . Assume first that  $\frac{3}{4} > s > \frac{1}{4}$ . We obtain

$$\begin{aligned}
|J| &\lesssim \sum_{j=1}^2 (\|\langle \alpha_j \psi_1, \psi_2 \rangle\|_{H^{-\frac{1}{4}+, \frac{1}{4}+}} \|\langle \psi_3, \psi_4 \rangle\|_{H^{-\frac{3}{4}-, -\frac{1}{4}-}} \\
&\quad + \|\langle \psi_1, \psi_2 \rangle\|_{H^{-\frac{1}{4}+, \frac{1}{4}+}} \|\langle \alpha_j \psi_3, \psi_4 \rangle\|_{H^{-\frac{3}{4}-, -\frac{1}{4}-}}).
\end{aligned}$$

By Theorem 1.3 with  $s_0 = \frac{1}{4}-$ ,  $b_0 = -\frac{1}{4}-$ ,  $s_1 = s_2 = s$ ,  $b_1 = b_2 = \frac{1}{2} + \epsilon$  for the first factors and  $s_0 = \frac{3}{4}+$ ,  $b_0 = \frac{1}{4}+$ ,  $s_1 = s$ ,  $s_2 = -s$ ,  $b_1 = \frac{1}{2} + \epsilon$ ,  $b_2 = \frac{1}{2} - 2\epsilon$  for the second factors we obtain under the assumption  $\frac{3}{4} > s > \frac{1}{4}$ :

$$|J| \lesssim \prod_{j=1}^3 \|\psi_j\|_{H^{s, \frac{1}{2}+\epsilon}} \|\psi_4\|_{H^{-s, \frac{1}{2}-2\epsilon}}.$$

Using the embedding  $X_{\pm}^{s,b} \subset H^{s,b}$  for  $s \in \mathbb{R}$  and  $b \geq 0$  we obtain the desired estimate.

Next assume  $s \geq \frac{3}{4}$ . We obtain

$$|J| \lesssim \sum_{j=1}^2 (\|\langle \alpha_j \psi_1, \psi_2 \rangle\|_{H^{s-1+, \frac{1}{4}+}} \|\langle \psi_3, \psi_4 \rangle\|_{H^{-s-, -\frac{1}{4}-}} + \|\langle \psi_1, \psi_2 \rangle\|_{H^{s-1+, \frac{1}{4}+}} \|\langle \alpha_j \psi_3, \psi_4 \rangle\|_{H^{-s-, -\frac{1}{4}-}}).$$

By Theorem 1.3 with  $s_0 = 1 - s-$ ,  $b_0 = -\frac{1}{4}-$ ,  $s_1 = s_2 = s$ ,  $b_1 = b_2 = \frac{1}{2} + \epsilon$  for the first factors and  $s_0 = s+$ ,  $b_0 = \frac{1}{4}+$ ,  $s_1 = s$ ,  $s_2 = -s$ ,  $b_1 = \frac{1}{2} + \epsilon$ ,  $b_2 = \frac{1}{2} - 2\epsilon$  for the second factors we obtain the same estimate as before.  $\square$

**Remark:** The potentials are completely determined by  $\psi$  and (4). We have  $A_\mu \sim |\nabla|^{-1} \langle \psi, \psi \rangle$ , so that for  $s < 1$ :

$$\|A_\mu\|_{\dot{H}^{2s}} \lesssim \|\langle \psi, \psi \rangle\|_{\dot{H}^{2s-1}} \lesssim \|\langle \psi, \psi \rangle\|_{L^{\frac{1}{1-s}}} \lesssim \|\psi\|_{L^{\frac{2}{1-s}}}^2 \lesssim \|\psi\|_{H^s}^2 < \infty$$

and

$$\|A_\mu\|_{\dot{H}^\epsilon} \lesssim \|\langle \psi, \psi \rangle\|_{\dot{H}^{\epsilon-1}} \lesssim \|\psi\|_{L^{\frac{4}{2-\epsilon}}}^2 \lesssim \|\psi\|_{H^s}^2 < \infty,$$

thus we obtain for  $0 < \epsilon \ll 1$  and  $s < 1$ :

$$A_\mu \in C^0([0, T], \dot{H}^{2s} \cap \dot{H}^\epsilon).$$

*Proof of Theorem 1.2.* We first show  $\psi_\pm \in X_{\pm}^{0,1}[0, T]$ . We have to prove

$$\|N(\psi_1, \psi_2, \psi_3)\|_{L_t^2([0, T], L_x^2)} \lesssim \prod_{j=1}^3 \|\psi_j\|_{L_t^\infty([0, T], H_x^{\frac{1}{3}})},$$

where the implicit constant may depend on  $T$ . This follows from the estimate

$$\begin{aligned} \||\nabla|^{-1} \langle \alpha_i \psi_j, \psi_k \rangle \psi_3\|_{L_x^2} &\lesssim \||\nabla|^{-1} \langle \alpha_i \psi_j, \psi_k \rangle\|_{L_x^6} \|\psi_3\|_{L_x^3} \lesssim \|\langle \alpha_i \psi_j, \psi_k \rangle\|_{L_x^{\frac{3}{2}}} \|\psi_3\|_{L_x^3} \\ &\lesssim \|\psi_j\|_{L_x^3} \|\psi_k\|_{L_x^3} \|\psi_3\|_{L_x^3} \lesssim \|\psi_j\|_{H_x^{\frac{1}{3}}} \|\psi_k\|_{H_x^{\frac{1}{3}}} \|\psi_3\|_{H_x^{\frac{1}{3}}}, \end{aligned}$$

and a similar estimate for the term  $\||\nabla|^{-1} \langle \psi_j, \psi_k \rangle \alpha_i \psi_3\|_{L_x^2}$ .

Assume now  $\psi \in C^0([0, T], H^{\frac{1}{3}+\epsilon})$ ,  $\epsilon > 0$ . Then we have shown that  $\psi_\pm \in X_{\pm}^{\frac{1}{3}+\epsilon, 0}[0, T] \cap X_{\pm}^{0,1}[0, T]$ . By interpolation we get  $\psi_\pm \in X_{\pm}^{\frac{1}{3}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}[0, T]$  for  $\epsilon \ll 1$ . Assume now that  $\psi, \psi' \in C^0([0, T], H^{\frac{1}{3}+\epsilon})$  are two solutions of (5), (6). Then we have

$$\begin{aligned} \sum_{\pm} \|\psi_\pm - \psi'_\pm\|_{X_{\pm}^{0, \frac{1}{2}+}[0, T]} &\lesssim T^{0+} \sum_{\pm} \|N(\psi, \psi, \psi) - N(\psi', \psi', \psi')\|_{X_{\pm}^{0, -\frac{1}{2}+}[0, T]} \\ &\lesssim T^{0+} \sum_{\pm, \pm_1, \pm_2, \pm_3} (\|N(\psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2}, \psi_{\pm_3})\|_{X_{\pm}^{0, -\frac{1}{2}+}[0, T]} \\ &\quad + \|N(\psi'_{\pm_1}, \psi_{\pm_2} - \psi'_{\pm_2}, \psi_{\pm_3})\|_{X_{\pm}^{0, -\frac{1}{2}+}[0, T]} \\ &\quad + \|N(\psi'_{\pm_1}, \psi'_{\pm_2}, \psi_{\pm_3} - \psi'_{\pm_3})\|_{X_{\pm}^{0, -\frac{1}{2}+}[0, T]}) \end{aligned} \quad (9)$$

Here  $\pm, \pm_j$  ( $j = 1, 2, 3$ ) denote independent signs. We want to show that for the first term the following estimate holds:

$$\begin{aligned} J &:= \int \langle N(\psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2}, \psi_{\pm_3}), \psi_4 \rangle dx dt \\ &\lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{X_{\pm_1}^{0, \frac{1}{2}+}} \|\psi_{\pm_2}\|_{X_{\pm_2}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}} \|\psi_{\pm_3}\|_{X_{\pm_3}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}} \|\psi_4\|_{X_{\pm_4}^{0, \frac{1}{2}-}}. \end{aligned} \quad (10)$$

We consider the case  $|\xi_0| \leq 1$  first. Similarly as in the proof of Theorem 1.1 we obtain

$$|J| \lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{X_{\pm_1}^{-\frac{1}{4}-\frac{\epsilon}{4}, \frac{1}{4}}} \|\psi_{\pm_2}\|_{X_{\pm_2}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}}} \|\psi_{\pm_3}\|_{X_{\pm_3}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}}} \|\psi_4\|_{X_{\pm_4}^{-\frac{1}{4}-\frac{\epsilon}{4}, \frac{1}{4}}},$$

which is more than sufficient. For  $|\xi_0| \geq 1$  we obtain

$$\begin{aligned} |J| &\lesssim \sum_{j=1}^2 (\|\langle \alpha_j(\psi_{\pm_1} - \psi'_{\pm_1}), \psi_{\pm_2} \rangle\|_{H^{-\frac{1}{2}, 0}} \|\langle \psi_{\pm_3}, \psi_4 \rangle\|_{H^{-\frac{1}{2}, 0}} \\ &\quad + \|\langle \psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2} \rangle\|_{H^{-\frac{1}{2}, 0}} \|\langle \alpha_j \psi_{\pm_3}, \psi_4 \rangle\|_{H^{-\frac{1}{2}, 0}}) \\ &\lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{H^{0, \frac{1}{2}+}} \|\psi_{\pm_2}\|_{H^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}} \|\psi_{\pm_3}\|_{H^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}} \|\psi_4\|_{H^{0, \frac{1}{2}-}}, \end{aligned}$$

where we used Theorem 1.3 for the first factor with the choice  $s_0 = \frac{1}{2}$ ,  $b_0 = 0$ ,  $s_1 = 0$ ,  $b_1 = \frac{1}{2} +$ ,  $s_2 = \frac{1}{4} + \frac{\epsilon}{4}$ ,  $b_2 = \frac{1}{4} + \epsilon$  and for the second factor with  $s_0 = \frac{1}{2}$ ,  $b_0 = 0$ ,  $s_1 = \frac{1}{4} + \frac{\epsilon}{4}$ ,  $b_1 = \frac{1}{4} + \epsilon$ ,  $s_2 = 0$ ,  $b_2 = \frac{1}{2} -$ . The embedding  $X_{\pm}^{s, b} \subset H^{s, b}$  for  $b \geq 0$  gives (10). The other terms in (9) are treated similarly. We obtain

$$\begin{aligned} &\sum_{\pm} \|\psi_{\pm} - \psi'_{\pm}\|_{X_{\pm}^{0, \frac{1}{2}+}[0, T]} \\ &\lesssim T^{0+} \sum_{j=1}^2 (\|\psi_{\pm_j}\|_{X_{\pm_j}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}[0, T]}^2 + \|\psi'_{\pm_j}\|_{X_{\pm_j}^{\frac{1}{4}+\frac{\epsilon}{4}, \frac{1}{4}+\epsilon}[0, T]}^2) \sum_{\pm} \|\psi_{\pm} - \psi'_{\pm}\|_{X_{\pm}^{0, \frac{1}{2}+}[0, T]}. \end{aligned}$$

For sufficiently small  $T$  this implies  $\|\psi_{\pm} - \psi'_{\pm}\|_{X_{\pm}^{0, \frac{1}{2}+}[0, T]} = 0$ , thus local uniqueness. By iteration  $T$  can be chosen arbitrarily.  $\square$

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